Satisfiability Modulo Abstraction for Separation Logic with Linked Lists

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Abstract

Separation logic is an expressive logic for reasoning about heap structures in programs. This paper presents a semi-decision procedure for deciding unsatisfiability of formulas in a fragment of separation logic that includes points-to assertions ($x \to y$), acyclic-list-segment assertions ($\text{ls}(x, y)$), logical-and, logical-or, separating conjunction, and septraction (the DeMorgan-dual of separating implication). The fragment that we consider allows negation at leaves, and includes formulas that lie outside other separation-logic fragments considered in the literature.

The semi-decision procedure is designed using concepts from abstract interpretation. The procedure uses an abstract domain of shape graphs to represent a set of heap structures, and computes an abstraction that over-approximates the set of satisfying models of a given formula. If the over-approximation is empty, then the formula is unsatisfiable.

We have implemented the method, and evaluated it on a set of formulas taken from the literature. The implementation is able to establish the unsatisfiability of formulas that cannot be handled by other existing approaches.

1. Introduction

Separation logic [33] is an expressive logic for reasoning about heap-allocated data structures in programs. It provides a mechanism for concisely describing program states by explicitly localizing facts that hold in separate regions of the heap. In particular, a “separating conjunction” ($\varphi_1 \land \varphi_2$) asserts that the heap can be split into two disjoint regions (“heaplets”) in which $\varphi_1$ and $\varphi_2$ hold, respectively [33]. A “septraction” ($\varphi_1 \land \neg \varphi_2$) asserts that a heaplet $h$ can be extended by a disjoint heaplet $h_1$ in which $\varphi_1$ holds, to create a heaplet $h_1 \cup h$ in which $\varphi_2$ holds [38]. The $\land \neg$ operator is sometimes called existential magic wand, because it is the DeMorgan-dual of the magic-wand operator “$\Rightarrow$” (also called separating implication); i.e., $\varphi_1 \land \neg \varphi_2$ iff $\neg(\varphi_1 \Rightarrow \neg\varphi_2)$.

The use of separation logic in manual, semi-automated, and automated verification tools is a burgeoning field [5, 14, 15, 19, 27]. Most of these incorporate some form of automated reasoning for separation logic, but only limited fragments of separation logic are typically handled.

This paper presents a semi-decision procedure for deciding the unsatisfiability of formulas in a fragment of separation logic that includes points-to assertions ($x \to y$), acyclic-list-segment assertions ($\text{ls}(x, y)$), empty-heap assertions ($\text{emp}$), and their negations; separating conjunction; septraction; logical-and; and logical-or. The fragment considered only allows negation at the leaves of a formula (§2.1), but still contains formulas that lie outside of previously considered fragments [4, 22, 25, 29, 30]. The semi-decision procedure can prove validity of implications of the form

$$\psi \Rightarrow (\varphi_1 \land \bigwedge_j \psi_j \Rightarrow \varphi_j),$$

(1)

where $\varphi_i$ and $\varphi_j$ are formulas that contain only $\land$, $\lor$, and positive or negative occurrences of $\text{emp}$, points-to, or $\text{ls}$ assertions; and $\psi$ and $\psi_j$ are arbitrary formulas in the logic fragment defined in §2.1. Consequently, we believe that ours is the first procedure that can prove the validity of formulas that contain both $\text{ls}$ and the magic-wand operator $\land \neg$. Furthermore, the semi-decision procedure is able to prove unsatisfiability of interesting classes of formulas that are outside of previously considered fragments, including (i) formulas that use conjunctions of separating-conjunctions with $\text{ls}$ or negations below separating-conjunctions, such as

$$(\text{ls}(a_1, a_2) \land \text{ls}(a_2, a_3)) \land (\neg\text{emp} \land \text{emp})$$

and (ii) formulas that contain both $\text{ls}$ and septraction ($\land \neg$), such as

$$(a_3 \Rightarrow a_1 \land \neg\text{ls}(a_1, a_4)) \land (a_3 \Rightarrow a_4 \land \neg\text{ls}(a_1, a_3)).$$

The former are useful for describing overlaid data structures; the latter are useful in dealing with interference effects when using rely/guarantee reasoning to verify programs with fine-grained concurrency [9, 38].

The key insight behind our approach is that the semi-decision procedure is designed using concepts from abstract interpretation [12]. Given a formula $\varphi$, the semi-decision procedure sets up an appropriate abstract domain that is tailored for representing information about the meanings of subformulas of $\varphi$. It uses an abstract domain of shape graphs [34] to represent a set of heap structures. The proof calculus that we present performs a bottom-up evaluation of $\varphi$, using a particular shape-graph interpretation. It computes an abstract value that over-approximates the set of satisfying models of $\varphi$. If the over-approximation is the empty set of shape graphs, then $\varphi$ is unsatisfiable. If $\varphi$ is satisfiable, then the procedure reports a set of abstract models.

This use of abstract domains to prove unsatisfiability places our work squarely in a recent line of research on using abstract values drawn from an abstract domain as a way to represent knowledge in implementations of decision procedures [16–18, 36, 37] (a technique we call “Satisfiability Modulo Abstraction”). Our work is the first to apply this idea to a fragment of separation logic.

One of the main advantages of the Satisfiability Modulo Abstraction approach is that it is able to reuse abstract-interpretation machinery to implement decision procedures. In [37], for instance, the polyhedral abstract domain—implemented in PPL [3]—is used

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to implement a decision procedure for the logic of linear rational arithmetic. In this paper, we use the abstract domain of shapes—implemented in TVLA [34]—in a novel way to implement a semi-
decision procedure for separation logic. The challenge was to in-
stantiate the parametric framework of TVLA to precisely represent
the literals and capture the spatial constraints of our fragment of
separation logic.

The nature of our semi-decision procedure is thus much dif-
ferent from other decision procedures for fragments of sepa-
ration logic.

The contributions of our work include the following:

• We show how a canonical-abstraction domain can be used to

overapproximate the set of models that satisfy a separation-logic

formula.

• We present rules for calculating the overapproximation of a

separation-logic formula for a fragment of separation logic that

consists of separating conjunction, sepstraction, logical-and, and

logical-or (§4).

• The semi-decision procedure is parameterized by a shape ab-

straction, and can be instantiated to handle (positive or nega-
tive) literals for points-to or acyclic-list-segment assertions—

which consist of a

store

and a

heaplet


The set of literals, denoted by

\( \text{Val Statelet} \)

and the unary predicate

\( \text{ls}(x, y) \)

if

\( (s, x) = s(y) \)

dom(h) = \emptyset

else there is a nonempty acyclic

path from \( s(x) \) to \( s(y) \) in \( h \), and

this path contains all heap cells in \( h \).

Figure 1: Satisfaction of an \( \mathcal{SL} \) formula \( \varphi \) with respect to statelet \( (s, h) \).

<table>
<thead>
<tr>
<th>Predicate</th>
<th>Intended Meaning</th>
</tr>
</thead>
<tbody>
<tr>
<td>( eq(v_1, v_2) )</td>
<td>Do ( v_1 ) and ( v_2 ) denote the same memory cell?</td>
</tr>
<tr>
<td>( q(v) )</td>
<td>Does pointer variable ( q ) point to memory cell ( v )?</td>
</tr>
<tr>
<td>( n(v_1, v_2) )</td>
<td>Does the ( n )-field of ( v_1 ) point to ( v_2 )?</td>
</tr>
</tbody>
</table>

Table 1: Core predicates used when representing states made up of acyclic

linked lists.

denotes the set of statelets that satisfy \( \varphi \): \( [\varphi] \triangleq \{(s, h) \mid (s, h) \models \varphi \} \).

2.2 2-Valued Logical Structures

We model full states—not statelets—by 2-valued logical structures. A logical structure provides an interpretation of a vocabulary

\( \text{Voc} = \{eq, \ldots, p_n\} \)

of predicate symbols (with given arities). \( \text{Voc}_k \) denotes the set of \( k \)-ary symbols.

DEFINITION 1. A 2-valued logical structure \( S \) over \( \text{Voc} \) is a pair

\( S = (U, \iota) \), where \( U \) is the set of individuals, and \( \iota \) is the inter-

pretation. Let \( \mathbb{B} = \{0, 1\} \) be the domain of truth values. For \( p \in \text{Voc}_k \),

\( \iota(p) : U^k \rightarrow \mathbb{B} \). We assume that \( eq \in \text{Voc}_2 \) is the identity relation:

(\( i \)) for all \( u \in U \), \( \iota(eq)(u, u) = 1 \), and (\( ii \)) for all \( u_1, u_2 \in U \) such

that \( u_1 \neq u_2 \) are distinct individuals, \( \iota(eq)(u_1, u_2) = 0 \).

The set of 2-valued logical structures over \( \text{Voc} \) is denoted by 2-

STRUCT[\text{Voc}].

A concrete state is modeled by a 2-valued logical structure over a

fixed vocabulary \( C \) of core predicates. Core predicates are part of

the underlying semantics of the linked structures that make up the

states of interest. Tab. 1 lists the core predicates that are used when

representing states made up of acyclic linked lists.

Without loss of generality, vocabularies exclude constant and

function symbols. Constant symbols can be encoded via unary

predicates, and \( n \)-ary functions via \( n + 1 \)-ary predicates. In both

cases, we need integrity rules—i.e., global constraints that restrict

the set of structures considered to the ones that we intend. The set

of unary predicates, \( \text{Voc}_1 \), always contains predicates that encode the

variables of the formula. In a minor abuse of notation, we overload

\( \ldots \) to denote both the name of variable \( x \) and the unary predicate

\( x(\cdot) \) that encodes the variable. The binary predicate \( n \in \text{Voc}_2 \)

codes link-node connections. In essence, the following integrity rules

restrict each \( x \in \text{Voc} \subseteq \text{Voc}_1 \) to serve as a constant, and restrict

relation \( n \) to encode a partial function:

\[
\begin{align*}
\text{for each } x \in \text{Voc}, & \forall v_1, v_2 : x(v_1) \land x(v_2) \Rightarrow eq(v_1, v_2) \\
\forall v_1, v_2, v_3 : n(v_3, v_1) \land n(v_3, v_2) \Rightarrow eq(v_1, v_2)
\end{align*}
\]
2.3 Connecting 2-Valued Logical Structures and SL Statelets

We use unary domain predicates, typically denoted by \(d, d', d_1, \ldots, d_k \in \text{Voc}_1\), to pick out regions of the heap that are of interest in the state that a logical structure models. The connection between 2-valued logical structures and SL statelets is formalized by means of the operation \(S_{\langle d, \cdot \rangle}\), which performs a projection of structure \(S\) with respect to a domain predicate \(d\):

\[
S_{\langle d, \cdot \rangle} = \left\{ (s, h) \mid s \in S, h = \{ (p, u) \mid p \in \text{Var}^S, u \in U^S, \text{and } p(u) \} \right\}
\]

(2)

The subscript \(\langle d, \cdot \rangle\) serves as a reminder that in Eqn. (3), only \(u_1\) needs to be in the region defined by \(d\). We lift the projection operation to apply to a set \(SS\) of 2-valued logical structures as follows: \(SS_{\langle d, \cdot \rangle} = \{ S_{\langle d, \cdot \rangle} \mid S \in SS \}\).

2.4 Representing Sets of SL Statelets using Canonical Abstraction

In the framework of Sagiv et al. [34] for logic-based abstract-interpretation, 3-valued logical structures provide a way to over-approximate possibly infinite sets of 2-valued structures in a finite way that can be represented in a computer. The application of Eqns. (2) and (3) to 3-valued structures means that the abstract-interpretation machinery developed by Sagiv et al. provides a finite way to overapproximate a possibly infinite set of SL statelets.

In 3-valued logic, a third truth value, denoted by 1/2, represents uncertainty. The set \(T = B \cup \{1/2\}\) of 3-valued truth values is partially ordered \(\sqsubseteq\) for \(i \in B\). The values 0 and 1 are definite values; 1/2 is an indefinite value.

**Definition 2.** A 3-valued logical structure \(S = \langle U, \iota \rangle\) is identical to a 2-valued structure, except that \(\iota\) maps each pair \(p \in \text{Voc}_1\), to a 3-valued function \(\iota(p): U \to T\). In addition, (i) for all \(u \in U\), \(\iota(eq)(u, u) \sqsupseteq 1\), and (ii) for all \(u_1, u_2 \in U\) such that \(u_1\) and \(u_2\) are distinct individuals, \(\iota(eq)(u_1, u_2) = 0\). (An individual \(u\) for which \(\iota(eq)(u, u) = 1/2\) is called a **symbolic individual**.)

The set of 3-valued logical structures over Voc is denoted by 3-STRUCT[Voc]. Note that 2-STRUCT[Voc] \(\subseteq\) 3-STRUCT[Voc].

As we will see below, a symbolic individual may represent more than one individual from certain 2-valued structures.

A 3-valued structure can be depicted as a directed graph with individuals as graph nodes (see Fig. 2). A symbolic individual is depicted with a double-ruled border. A unary predicate \(p \in \text{Var}\) is represented in the graph by having an arrow from the predicate name \(p\) to all nodes of individuals \(u\) for which \(\iota(p)(u) \sqsupseteq 1\). An arrow between two nodes indicates that a binary predicate holds for the corresponding pair of individuals. (To reduce clutter, in the figures in this paper, the only binary predicate shown is the predicate \(n \in \text{Voc}_2\).) A predicate value of 1/2 is indicated by a dotted arrow, a value of 1 by a solid arrow, and a value of 0 by the absence of an arrow. A unary predicate \(p \in \text{Voc}_1 - \text{Var}\) is listed, with its value, inside the node of each individual \(u\) for which \(\iota(p)(u) \sqsupseteq 1\). A nullary predicate is displayed in a rectangular box.

To define a suitable abstraction of 2-valued logical structures, we start with the notion of structure embedding [34]:

**Definition 3.** Given \(S = \langle U, \iota \rangle\) and \(S' = \langle U', \iota' \rangle\), two 3-valued structures over the same vocabulary Voc, and \(f: U \to U'\), a surjective function, \(f\) *embeds* \(S\) in \(S'\), denoted by \(S \sqsubseteq S'\), if for all \(p \in \text{Voc}\) and \(u_1, \ldots, u_k \in U\),

\[
\iota'(p)(u_1, \ldots, u_k) = \bigcup_{u_1, \ldots, u_k \in U, f(u_1) = u'_1, \ldots, f(u_k) = u'_k} \iota(p)(u_1, \ldots, f(u_k))
\]

then \(S'\) is the tight embedding of \(S\) with respect to \(f\), denoted by \(S' = f(S)\). (Note that we overload \(f\) to also mean the mapping on structures \(f: 3-\text{STRUCT}[\text{Voc}] \to 3-\text{STRUCT}[\text{Voc}]\) induced by \(f: U \to U'\).

Intuitively, \(f(S)\) is obtained by merging individuals of \(S\) and by defining the valuation of predicates accordingly (in the most precise way). The relation \(\sqsubseteq\), which will be denoted by \(\sqsubseteq\), is the natural information order between structures that share the same universe. One has \(S \sqsubseteq S' \iff f(S) \sqsubseteq S'\). Henceforth, we use \(S \sqsubseteq S'\) to mean "there exists a surjective \(f: U \to U'\) such that \(f(S) \sqsubseteq S'\)."

However, embedding alone is not enough. The challenge for representing and manipulating sets of 2-valued structures is that the universe of a structure is of a *priori* unbounded size. Consequently, we need a method that, for a 2-valued structure \(\langle U, \iota \rangle \in 2-\text{STRUCT}[\text{Voc}]\), abstracts \(U\) to an abstract universe \(U'\) of bounded size. The idea behind canonical abstraction [34, §4.3] is to choose a subset \(A \subseteq \text{Voc}_1\) of abstraction predicates, and to define an equivalence relation \(\approx_{A, S}\) on \(U\) that is parametrized by the logical structure \(S = \langle U, \iota \rangle \in 2-\text{STRUCT}[\text{Voc}]\) to be abstracted:

\[
u_1 \approx_{A, S} \nu_2 \iff \forall \nu \in A: \iota(p)(\nu_1) = \iota(p)(\nu_2).
\]

This equivalence relation defines the surjective function \(f^S_2: U \to (U / \approx_{A, S})\), which maps an individual to its equivalence class. We thus have the Galois connection

\[
\varphi(2-\text{STRUCT}[\text{Voc}]) \overset{\approx_{2}}{\to} \varphi(3-\text{STRUCT}[\text{Voc}])
\]

\[
\alpha(X) = \{ f^S_2(S) \mid S \in X \} \quad \gamma(Y) = \{ S \mid S^2 \subseteq Y \land S \sqsubseteq S'\}
\]

where \(f^S_2\) in the definition of \(\alpha\) denotes the tight-embedding function for logical structures induced by the node-embedding function \(f^S_2: U \to (U / \approx_{A, S})\). The abstraction function \(\alpha\) is referred to as the canonical abstraction. Note that there is an upper bound on the size of each structure \(U^2, \iota^2\) in 3-STRUCT[Voc] that is in the image of \(\alpha\): \(|U^2| \leq 2^{|A|}\)—and thus the power-set of the image of \(\alpha\) is a finite sublattice of \(\varphi(3-\text{STRUCT}[\text{Voc}])\).

For technical reasons, it turns out to be convenient to work with 3-valued structures other than the ones in the image of \(\alpha\): however, we still want to restrict ourselves to a finite sublattice of \(\varphi(3-\text{STRUCT}[\text{Voc}])\). With this motivation, we make the following definition [2]:

**Definition 4.** A 3-valued structure \(\langle U^2, \iota^2 \rangle \in 3-\text{STRUCT}[\text{Voc}]\) is bounded (with respect to abstraction predicates \(A\)) if for every \(u_1, u_2 \in U^2\), where \(u_1 \neq u_2\), there exists an abstraction predicate symbol \(p \in \text{Voc}_1\) such that \(\iota^2(p)(u_1) = 0\) and \(\iota^2(p)(u_2) = 1\), or \(\iota^2(p)(u_1) = 1\) and \(\iota^2(p)(u_2) = 0\). B-STRUCT[Voc, A] denotes the set of such structures.

Dfn. 4 also imposes an upper bound on the size of a structure \(\langle U^2, \iota^2 \rangle \in B-\text{STRUCT}[\text{Voc}, A]\)—again, \(|U^2| \leq 2^{|A|}\)—and thus \(\varphi(B-\text{STRUCT}[\text{Voc}, A])\) is a finite sublattice of \(\varphi(3-\text{STRUCT}[\text{Voc}])\). It defines the abstract domain that we use, the abstract domain whose elements are subsets of B-STRUCT[Voc, A], which will be denoted by \(A[A, \text{Voc}, A]\) (for brevity, we call such a domain a "canonical-abstraction domain", and denote it by \(A\) when Voc and \(A\) are understood.) The Galois connection we work with is thus

\[
\varphi(2-\text{STRUCT}[\text{Voc}]) \overset{\approx_{2}}{\to} \varphi(B-\text{STRUCT}[\text{Voc}, A]) = A[A, \text{Voc}, A]
\]

\[
\alpha(X) = \{ f^S_2(S) \mid S \in X \} \quad \gamma(Y) = \{ S \mid S^2 \subseteq Y \land S \sqsubseteq S'\}
\]

The ordering on \(\varphi(B-\text{STRUCT}[\text{Voc}, A]) = A[A, \text{Voc}, A]\) is the Hoare ordering: \(S_1 \sqsubseteq S_2\) if for all \(s_1 \in S_1\) there exists \(s_2 \in S_2\) such that \(s_1 \sqsubseteq s_2\).
ϕ used to demarcate the heaplets that must satisfy $\gamma$ such that a decision procedure using a formula that is unsatisfiable over acyclic heaps: $x \mapsto y \cdot y \mapsto x$. App. A illustrates the procedure using a formula that is satisfiable over acyclic heaps: $x \mapsto y \cdot y \mapsto x$. App. A illustrates the procedure using a formula that is satisfiable over acyclic heaps: $x \mapsto y \cdot y \mapsto x$.

Consider $\phi \equiv x \mapsto y \cdot y \mapsto x$. We want to compute $A \in \mathcal{A}$ such that $\gamma(A)_{(d,\cdot)} \geq \phi$. The key to handling the $\ast$ operator is to introduce two new domain predicates $d_1$ and $d_2$, which are used to demarcate the heaplets that must satisfy $\phi \equiv x \mapsto y$ and $\varphi_2 \equiv y \mapsto x$, respectively. We have designed $\mathcal{A}$ so that there exist $A_1, A_2 \in \mathcal{A}$ such that $\gamma(A_1)_{(d_1,\cdot)} = [x \mapsto y]$ and $\gamma(A_2)_{(d_2,\cdot)} = [y \mapsto x]$, respectively. Tab. 2 describes the abstraction predicates we use, $A_1$ and $A_2$ each consist of a single 3-valued structure, shown in Fig. 2(b) and Fig. 2(c), respectively. Furthermore, to satisfy $\varphi_1 \ast \varphi_2$, $d_1$ and $d_2$ are required to be disjoint regions whose union is $d$. $\mathcal{A}$ also contains an abstract value, which we will call $D$, that represents this disjointness constraint exactly. $D$ consists of four 3-valued structures. Fig. 2(a) shows the “most general” of them; it represents two disjoint regions, $d_1$ and $d_2$, that partition the $d$ region (where each of $d_1$ and $d_2$ contain at least one cell). The summary individual labeled $\neg d, \neg d_1, \neg d_2$ in Fig. 2(a) represents a region that is disjoint from $d$. (See also Fig. 5.)

Note that here and throughout the paper, for brevity the figures only show predicates that are relevant to the issue under discussion.

**Meet for a Canonical-Abstraction Domain.** To impose a necessary condition for $x \mapsto y \cdot y \mapsto x$ to be satisfiable, we take the meet of $D, A_1,$ and $A_2: [x \mapsto y \cdot y \mapsto x] \subseteq D \cap A_1 \cap A_2$. Figs. 2(d), (e), and (f) show some of the structures that arise in $D \cap A_1 \cap A_2$.

The meet operation in $\mathcal{A}$ is defined in terms of the greatest-lower-bound operation induced by the approximation order in the lattice $\text{B-STRUCT}[\text{Voc, } \mathcal{A}]$. Arnold et al. [2] show that in general this operation is NP-complete; however, they define an algorithm based on graph matching that typically performs well in practice [23, §8.3]. To understand some of the subtleties of meet, consider Fig. 2(d), which shows one of the structures in $D \cap A_1$ (i.e., $\mathcal{A}$). Fig. 2(a) and Fig. 2(b)).

- From the standpoint of Fig. 2(b), meet caused the summary individual labeled “$\neg d_1$” to be split into two summary individuals: “$\neg d, \neg d_1, \neg d_2$” and “$d, \neg d_1, d_2$.”
- From the standpoint of Fig. 2(a), meet caused the summary individual labeled “$d, d_1, \neg d_2$” to (i) become a non-summary individual, (ii) acquire the value 1 for $x$, $r[n,x]$, and $\text{next}[n,y]$, and (iii) acquire the value 0 for $y$ and $r[n,y]$.

Fig. 2(e) shows one of the structures in $(D \cap A_1) \cap A_2$, i.e., Fig. 2(d) and Fig. 2(c), which causes further (formerly indefinite) elements to acquire definite values.

Arnold et al. develop a graph-theoretic notion of the possible correspondences among individuals in the bounded structures that are arguments to meet, and structure the meet algorithm around the set of possible correspondences [2, §4.2].

**Improving Precision Using Semantic-Reduction Operators.** Fig. 2(e) still contains a great deal of indefinite information because the meet operation does not take into account the integrity constraints on structures. For instance, for the structures that we use to represent states and STL statelets, we use a unary predicate $\text{next}[n,y]$, which holds for individuals whose $n$-link points to the
Table 2: Voc consists of the predicates shown above, together with the ones in Tab. 1. All unary predicates are abstraction predicates; that is, \( A = \text{Voc} \).

\[
\forall v_1, v_2. \text{next}[n, y](v_1) \land y(v_2) \implies n(v_1, v_2).
\]

(4)

In particular, in Fig. 2(e) the individual pointed to by \( x \) has \( \text{next}[n, y] = 1 \); however, the edge to the individual pointed to by \( y \) has the value 1/2. Similarly, we force the semi-decision procedure to consider only acyclic heaps by imposing the integrity constraint

\[ \exists v_1, v_2. \text{next}[n, v_1](v_2) \land \text{t}[n](v_2, v_1). \]

To improve the precision of the (graph-theoretic) meet, the semantic-reduction rule would find an irreconcilable inconsistency in Fig. 2(f): the first two predicate values mean that \( u_2 \) is reachable from the individual pointed to by \( x \) along \( \text{n-links} \), which contradicts \( \text{r}[n, x](u_2) = 0 \).

The operation that applies type-1 and type-2 rules until no more changes are possible is called \( \text{coerce} \) (because it coerces \( X_S \) to a better representation \( X' \)). Sagiv et al. [34, §6.4] and Bogudlov et al. [6, 7] discuss algorithms for \( \text{coerce} \).
The abstract value for $x$ atoms $A$ Fig. 4(b) is the single structure in possible to represent the positive literals $A$ following observation: if the individuals in $d_1$ are disjoint from $d_3$, and that the individuals in $d$ are the disjoint union of the individuals in $d_1$ and $d_2$. With only a slight abuse of notation, the meaning of $[d = d_1 \cdot d_2]^2$ can be expressed as follows:

$$\gamma([d = d_1 \cdot d_2]^2)|_{(d_1, \cdot)} \supseteq [\varphi_1]$$ and $\gamma(A_2)|_{(d_2, \cdot)} \supseteq [\varphi_2]$. \hspace{1cm} (5)

$[d = d_1 \cdot d_2]^2 \in A$ is used to express the constraint that the individuals in $d_1$ are disjoint from $d_3$, and that the individuals in $d$ are the disjoint union of the individuals in $d_1$ and $d_2$. With only a slight abuse of notation, the meaning of $[d = d_1 \cdot d_2]^2$ can be expressed as follows:

$$\gamma([d = d_1 \cdot d_2]^2)|_{(d_1, \cdot)} \supseteq \{(s, h, h_1, h_2) \mid h_1 \# h_2$$ and $h_1 \cdot h_2 = h \}.$$ \hspace{1cm} (6)

Fig. 5 shows the four structures in the abstract value $[d_1 = d_3 \cdot d_4]^2$, where $d_1$, $d_3$, and $d_4$ are domain predicates.

- Fig. 4(b) represents a singleton list from $x$ to $y$. That is, $x \neq y$ and $x \neq y$, and for all individuals $v$ in $d$, $v$ is reachable from $x$ and $\text{link}[d, n, y](v)$ is true. (See line 6 of Tab. 2.)
- Fig. 4(c) represents acyclic linked lists of length two or more from $x$ to $y$.

Fig. 4(b) is the single structure in $A_{x \rightarrow y}$. The abstract values for atoms $x = y$, true, and emp are straightforward. We see that it is possible to represent the positive literals true, emp, $x = y$, $x \rightarrow y$, and $\text{ls}(x, y)$ precisely in $A$; that is, we have $\gamma(A_1)|_{(d_1)} = [\top]$. Furthermore, because the canonical-abstraction domain $A$ is closed under negation [24, 40], we are able to represent the negative literals $x \neq y$, $\neg\text{true}$, $\neg\text{emp}$, $\neg\text{ls}(x, y)$, and $\neg x \rightarrow y$ precisely in $A$, as well.

The rest of the rules in Fig. 3 can be derived by reinterpreting the concrete logical operators using an appropriate abstract operator. In particular, logical-and is reinterpreted as meet, and logical-or are straightforward. The ($\land$)-rule and ($\lor$)-rule are straightforward. The ($\land$)-rule and ($\lor$)-rule are justified by the following observation: if $\gamma(A_1)|_{(d_1)} \supseteq [\varphi_1]$ and $\gamma(A_2)|_{(d_2)} \supseteq [\varphi_2]$, then $\gamma(A_1 \land A_2)|_{(d_1 \cdot d_2)} \supseteq [\varphi_1 \land \varphi_2]$ and $\gamma(A_1 \lor A_2)|_{(d_1 \cdot d_2)} \supseteq [\varphi_1 \lor \varphi_2]$.

For a given structure $A = (U, i)$ and unary domain predicate $d_i$, we use the phrase “individuals in $d_i$” to mean the set of individuals $\{u \in U \mid i(d_i)(u) = 1\}$.

The ($\ast$)-rule computes $A \in A$ such that $\gamma(A)|_{(d_1)} \supseteq [\varphi_1 \ast \varphi_2]$. The handling of separating conjunction $\varphi_1 \ast \varphi_2$ is based on the following insights:

The domain predicates $d_1$ and $d_2$ are used to capture the heapelets $h_1$ and $h_2$ that satisfy $\varphi_1$ and $\varphi_2$, respectively. That is,

$$\gamma(A)|_{(d_1)} \supseteq [\varphi_1]$$ and $\gamma(A_2)|_{(d_2, \cdot)} \supseteq [\varphi_2]$. \hspace{1cm} (5)

Using Eqns. (5) and (6) in the definition of $\varphi_1 \ast \varphi_2$, we have

$$[\varphi_1 \ast \varphi_2] = \{(s, h_1, h_2, h_1 \# h_2 \text{ and } h_1 \cdot h_2 = h \mid \varphi_1$$ and $\varphi_2\} \subseteq ([d = d_1 \cdot d_2]^2 \cap A_1 \cap A_2) \epsilon d^d$$

The handling of sepration logic is similar to the handling of separating conjunction in the ($\ast$)-rule, except for the condition that $h_2 = h \cdot h_1$. This requirement is easily handled by using $[d_2 = d_1 \cdot d_1]^2$. App. A illustrates the application of the ($\ast$)-rule.

**Theorem 1.** The rules in Fig. 3 are sound; that is, if the rules in Fig. 3 say that $\varphi \models d \models A$, then $\gamma(A)|_{(d_1)} \subseteq [\varphi]$. \hspace{1cm} $\square$

The proof follows from the fact that each of the abstract operators is sound.

**Discussion.** As discussed in [31, §4], there exist no methods that handle negations below a separating conjunction. Our fragment of separation logic admits negations at the leaves of formulas, and, thus, is the first approach that can handle formulas with negations below a separating conjunction.

It is, however, non-trivial to extend our technique to handle general negation. Let $(\cdot)^\#$ denote the set-complementation operation. Let $\neg^\#(\cdot)$ denote the abstract negation operation; that is, $\gamma(\neg^\#(A)) \supseteq \gamma(A)^\#$, and $\neg^\#(A) \supseteq \alpha(\gamma(A))^\#$. Suppose that $\gamma(A)|_{(d_1)} \subseteq [\varphi_1]$; in general, $\gamma(\neg^\#(A))|_{(d_1)}$ is not guaranteed to overapproximate the models of $\neg^\#\varphi$. Furthermore, it is non-trivial to extend our technique to prove validity of general implications. Suppose that we would like to prove the validity of $\varphi_1 \Rightarrow \varphi_2$, where $\varphi_1, \varphi_2 \in SL$. Let $A_1$ approximate the set of models of $\varphi_1$, and $A_2$ approximate the set of models of $\varphi_2$. $A_1 \subseteq A_2$ does not imply $[\varphi_1] \subseteq [\varphi_2]$.

### 5. Experimental Evaluation

This section presents the results of our experiments to evaluate the costs and benefits of our approach. The implementation and benchmarks can be accessed (anonymously) at [1]. The experiments were designed to shed light on the following questions:

1. How costly is the semi-decision procedure (in terms of time)?
2. How often is the semi-decision procedure able to determine that a formula is unsatisfiable?
Experiments were run on a single core of a 2-process or, explicit use of the meet operator [2] for a canonical-abstrac
tion formulas:

Table 3: Number of formulas that contain each of the 8L operators in the three groups of formulas used in the experiments. The columns labeled “+” and “−” indicate the number of atoms occurring as positive and negative literals, respectively.

<table>
<thead>
<tr>
<th>Group</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>8L</td>
<td>577</td>
<td>245</td>
<td>8</td>
<td>8</td>
</tr>
</tbody>
</table>

Table 4: Unsatisfiable formulas. A ✓ in the U-column indicates that the semi-decision procedure was able to prove the formula unsatisfiable; a ? indicates that the semi-decision procedure was not able to prove the formula unsatisfiable. The time is in seconds.

<table>
<thead>
<tr>
<th>Formula</th>
<th>U</th>
<th>Time</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1) a1 → a2 ∧ ¬b(a1, a2)</td>
<td>✓</td>
<td>1.41</td>
</tr>
<tr>
<td>(2) a1 → a2 ⊗ a2 → a1</td>
<td>✓</td>
<td>1.68</td>
</tr>
<tr>
<td>(3) ¬emp ∧ (b(a1, a2) ∧ b(a2, a1))</td>
<td>✓</td>
<td>2.04</td>
</tr>
<tr>
<td>(4) a1 ≠ a2 ∧ (b(a1, a2) ∧ b(a2, a1))</td>
<td>✓</td>
<td>1.91</td>
</tr>
<tr>
<td>(5) b(a1, a2) ∧ b(a2, a3) ∧ ¬b(a1, a3)</td>
<td>✓</td>
<td>3.75</td>
</tr>
<tr>
<td>(6) b(a1, a2) ∧ emp ∧ a1 ≠ a2</td>
<td>✓</td>
<td>1.41</td>
</tr>
<tr>
<td>(7) (a1 → a2 → true) ∧ (a2 → a3 → true) ∧ (true → a3 → a1)</td>
<td>?</td>
<td>4.34</td>
</tr>
<tr>
<td>(8) (a1 → a2 ⊗ true) ∧ (a1 → a2 → true)</td>
<td>?</td>
<td>2.32</td>
</tr>
<tr>
<td>(9) b(a1, a2) → ¬b(a2, a3) ∧ b(a1, a3)</td>
<td>✓</td>
<td>6.50</td>
</tr>
<tr>
<td>(10) b(a1, a2) ∧ b(a1, a3) ∧ ¬emp ∧ a2 ≠ a3</td>
<td>✓</td>
<td>1.91</td>
</tr>
<tr>
<td>(11) (b(a1, a2) ∧ true → a3) ∧ (true ∧ b(a2, a1) ∧ a2 ≠ a1)</td>
<td>?</td>
<td>30.6</td>
</tr>
<tr>
<td>(12) (a1 → a2 → b(c1, c2)) ∧ (a2 → a3 ∧ ¬emp) ∧ (a3 → a1 ∧ a5 ∧ a6 ∧ true)</td>
<td>?</td>
<td>40.5</td>
</tr>
<tr>
<td>(13) ¬emp ∧ (a1 = n1 ∧ a1 → e1 ∧ ((a1 → e1 ∧ e1 = n1) ∧ true)) ∧ b(c1, c2)</td>
<td>?</td>
<td>3.81</td>
</tr>
<tr>
<td>(14) (b(b(a1, a2) ∧ a1 ≠ a2) ∧ b(a2, a3) ∧ a2 ≠ a3) ∧ (b(a1, a2) ∧ a1 ≠ a2) ∧ (b(a2, a3) ∧ a2 ≠ a3) ∧ (b(a1, a2) ∧ a1 ≠ a2) ∧ (b(a2, a3) ∧ a2 ≠ a3) ∧ (true ∧ (a1 → e1 ∧ true))</td>
<td>✓</td>
<td>11.3</td>
</tr>
<tr>
<td>(15) b(a1, a2) → ¬b(a1, a2) ∧ ¬emp</td>
<td>✓</td>
<td>1.98</td>
</tr>
<tr>
<td>(16) (a3 → a4 → b(c1, a1) ∧ a4 ∧ ¬b(a1, a3))</td>
<td>✓</td>
<td>2.09</td>
</tr>
<tr>
<td>(17) ((a2 → a3 ∧ ¬b(a2, a4)) ∧ ¬b(a1, a3))</td>
<td>✓</td>
<td>3.89</td>
</tr>
<tr>
<td>(18) (a2 → a3 → ¬b(a2, a4)) ∧ ¬b(a3, a1) ∧ a2 = a4</td>
<td>✓</td>
<td>3.87</td>
</tr>
<tr>
<td>(19) (a1 → a2 → b(a1, a3)) ∧ ¬b(a2, a3) ∧ (true ∧ (a1 → e1 ∧ true))</td>
<td>✓</td>
<td>3.52</td>
</tr>
<tr>
<td>(20) (b(b(a1, a2) ∧ a1 ≠ a2) → ¬b(c1, c2)) ∧ e1 ≠ a1 ∧ e2 ≠ a2 ∧ ¬b(c1, a1)</td>
<td>?</td>
<td>4.56</td>
</tr>
<tr>
<td>(21) a1 ≠ a4 ∧ (b(a1, a4) → ¬b(c1, e2)) ∧ a4 = e2 ∧ ¬b(c1, a1)</td>
<td>?</td>
<td>5.62</td>
</tr>
<tr>
<td>(22) (b(b(a1, a2) ∧ a1 ≠ a2) → ¬b(c1, e2)) ∧ e2 ≠ a2 ∧ e1 ≠ a1 ∧ ¬b(a2, c2)</td>
<td>?</td>
<td>4.65</td>
</tr>
<tr>
<td>(23) ((a2 → a3 → ¬b(a2, a4)) → ¬b(a3, a1)) ∧ ¬b(a4, a1) ∨ a2 = a4</td>
<td>?</td>
<td>4.24</td>
</tr>
</tbody>
</table>

3. For unsatisfiable formulas that are beyond the capabilities of other existing tools, is the semi-decision procedure actually able to determine that the formulas are unsatisfiable?

**Setup.** The semi-decision procedure is written in OCaml; it compiles a formula to a proof DAG, expressed as an equation system. The abstract-value manipulations in the proof rules of Fig. 3 are performed using ITLVA, a modified version of TVLA [26] that was implemented for performing interprocedural shape analysis [23, 88]. ITLVA (i) replaces TVLA's notion of an intraprocedural control-flow graph by the more general notion of equation system, in which transfer functions may depend on more than one argument, and (ii) supports a more general language in which to specify equation systems. In particular, the ITLVA language supports explicit use of the meet operator [2] for a canonical-abstraction domain. Experiments were run on a single core of a 2-processor, 4-core-per-processor 2.27 GHz Xeon computer running Red Hat Linux 6.5.

**Test Suite.** Our test suite consists of three groups of unsatisfiable formulas:

- **Group 1**, shown in Tab. 4, was chosen to evaluate our procedure on a wide spectrum of formulas.
- **Group 2** was created by replacing the Boolean variables a and b in the template $T_1 \equiv \sim a \land emp \land (a \land b)$ with the 8 literals $\text{ls}$ of $\text{SL}$ that is, $\text{true}$, $\text{emp}$, $x \mapsto y$, $\text{ls}(x, y)$, and their negations. Five of the 64 instantiations of template $T_2$ are shown in Tab. 5. The 512 instantiations of template $T_2$ are shown in Tab. 6.
- **Group 3** was created by replacing the Boolean variables a, b, and c in the template $T_2 \equiv \text{emp} \land a \land (b \land c \land \sim \text{emp} \land \sim a)$ with the 8 literals $\text{ls}$ of $\text{SL}$. Five of the 512 instantiations of template $T_2$ are shown in Tab. 6.

Templates $T_1$ and $T_2$ are based on work by Hou et al. [22] on Boolean separation logic. Templates $T_1$ and $T_2$ are listed as formulas 15 and 19, respectively, in [22, Tab. 2]. In total, there were 599 formulas in our test suite. Tab. 3 summarizes the characteristics of the corpus based on the occurrences of the $\text{SL}$ operators.

Though not shown in this section, we also evaluated our procedure on a set of satisfiable formulas. The procedure reports a set of abstract models when given a satisfiable formula. The time taken to compute these abstract models is similar to that for proving formulas unsatisfiable.

We now answer Questions 1–3 posed at the beginning of this section using the three groups of formulas.

**Group 1 Results.** The running time of our procedure on the formulas listed in Tab. 4 was often on the order of five seconds. The procedure was able to prove unsatisfiability for all formulas, except (23). We believe that formulas (9)–(23) are beyond the scope of existing tools. Formulas (9)–(14) demonstrate that we can handle formulas that describe overlapping data structures, including conjunctions of separating conjunctions. Formulas (15)–(21) demonstrate that we can handle formulas that contain occurrences of both $\text{ls}$ and separation.
Table 5: Example instantiations of $T_1 \overset{df}{=} \neg a \land \emp \land (a \circledast b)$, where $a, b \in \text{lits}$. A $\checkmark$ in the U-column indicates that the semi-decision procedure was able to prove the formula unsatisfiable. The time is in seconds.

<table>
<thead>
<tr>
<th>Formula</th>
<th>U Time</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1) $(\neg(a \rightarrow a_2) \land \emp \land (a_1 \rightarrow a_2 \land a_3 \rightarrow a_4))$</td>
<td>$\checkmark$ 3.40</td>
</tr>
<tr>
<td>(2) $a \rightarrow a_2 \land \emp \land (a_1 \rightarrow a_2 \land a_3 \rightarrow a_4)$</td>
<td>$\checkmark$ 4.97</td>
</tr>
<tr>
<td>(3) $(\neg(a_1 \rightarrow a_2) \land \emp \land (a_1 \rightarrow a_3 \land a_2 \rightarrow a_4))$</td>
<td>$\checkmark$ 5.64</td>
</tr>
<tr>
<td>(4) $b(a_1, a_2) \land \emp \land (\neg b(a_1, a_2) \land b(a_3, a_4))$</td>
<td>$\checkmark$ 11.3</td>
</tr>
<tr>
<td>(5) $(\neg b(a_1, a_2) \land \emp \land (b(a_1, a_2) \land b(a_3, a_4)))$</td>
<td>$\checkmark$ 13.5</td>
</tr>
</tbody>
</table>

Table 6: Example instantiations of $T_2 \overset{df}{=} \emp \land a \land (b \circledast c \rightarrow \neg \circledast \land \emp \land \neg (a \rightarrow a_2)))$, where $a, b, c \in \text{lits}$. A $\checkmark$ in the U-column indicates that the semi-decision procedure was able to prove the formula unsatisfiable. The time is in seconds.

<table>
<thead>
<tr>
<th>Formula</th>
<th>U Time</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1) $\emp \land b(a_1, a_2) \land (b(a_3, a_4) \land (b(a_5, a_6) \rightarrow \neg \circledast \land \emp \land \neg (a_1 \rightarrow a_2)))$</td>
<td>$\checkmark$ 6.07</td>
</tr>
<tr>
<td>(2) $\emp \land \neg \circledast \land (b(a_3, a_4) \land (a_5 \rightarrow a_6) \rightarrow \circledast \land \emp \land \neg (a_1 \rightarrow a_2)))$</td>
<td>$\checkmark$ 3.34</td>
</tr>
<tr>
<td>(3) $\emp \land a_1 \rightarrow a_2 \land (a_3 \rightarrow a_4 \land (a_5 \rightarrow a_6) \rightarrow \circledast \land \neg (a_1 \rightarrow a_2)))$</td>
<td>$\checkmark$ 3.79</td>
</tr>
<tr>
<td>(4) $\emp \land \neg b(a_1, a_2) \land (\neg b(a_3, a_4) \land (b(a_5, a_6) \rightarrow \neg \circledast \land \emp \land b(a_1, a_2)))$</td>
<td>$\checkmark$ 8.05</td>
</tr>
<tr>
<td>(5) $\emp \land b(a_1, a_2) \land (b(a_3, a_4) \land (b(a_5, a_6) \rightarrow \neg \circledast \land \emp \land b(a_1, a_2)))$</td>
<td>$\checkmark$ 8.10</td>
</tr>
</tbody>
</table>

The literature related to reasoning about separation logic is vast, and we mention only a small portion of it in this section. Decidability results related to first-order separation logic are discussed in [8, 10]. A fragment of separation logic for which it is decidable to check validity of entailments was introduced by Berdine et al. [4]. The fragment includes points-to and linked-list predicates, but no separation, or negations of points-to or linked-list predicates. More recent approaches deal with fragments of separation logic that are incomparable to ours [22, 25, 29]; in particular, none of the latter papers handle linked lists. We based our experiments on formulas listed in Hou et al.’s work on Boolean separation logic [22]—the only paper we found that listed formulas outside the syntactic fragment defined by Berdine et al. We believe that our technique represents the first important step in designing a verification system that uses a richer fragment of separation logic.

Most approaches to separation-logic reasoning use a syntactic proof-theoretic procedure [4, 30]. Two exceptions are the approaches of Cook et al. [11] and Enea et al. [20], which use a more semantics-based approach: they represent separation-logic formulas as graphs in a particular normal form, and then prove that one formula entails another by finding a homomorphism between the corresponding graphs. Our approach is also semantics-based, but has more of an algebraic flavor: our method performs a bottom-up evaluation of a formula $\varphi$ using a particular shape-analysis interpretation (Fig. 3); if the answer is the empty set of 3-valued structures, then $\varphi$ is unsatisfiable.

To deal with overlaid data-structures, Enea et al. [20] introduce the $\ast_w$ operator: the $\ast_w$ operator specifies data structures that share sets of objects as long as they are built over disjoint sets of fields. Their approach, however, does not handle conjunctions of separating conjunctions or negations of the $\ls$-predicate. Thus, [20] cannot handle formulas (9)–(14) in Tab. 4, even though these formulas do not contain sepration. Note that, for instance, the logical conjunction in formula (9) cannot be replaced by the $\ast_w$ operator.

Piskac et al. [31] present a decision procedure for a decidable fragment of separation logic based on a reduction to a particular decidable first-order theory. Unlike our approach, the approach in [31] does not handle sepration or negations below a separating conjunction.

The explicit use of abstract values drawn from an abstract domain as a way to represent knowledge in implementations of decision procedures is a technique that has been receiving increased attention of late [16–18, 36, 37]. As far as we know, our work is the first to apply this idea to a fragment of separation logic.

Many researchers pigeonhole TVLA [26] as a system exclusively tailored for “shape analysis”. In fact, it is actually a metasystem for (i) defining a family of logical structures 2-STRUCT[$\text{Voc}$], and (ii) defining canonical-abstraction domains whose elements represent sets of 2-STRUCT[$\text{Voc}$]. The ITVLA [23, §8] variant of TVLA is a different packaging of the classes that make up the TVLA implementation, and demonstrates better that canonical abstraction is a general-purpose method for abstracting the structures that are a logic’s domain of discourse.

To simplify matters, the separation-logic fragment addressed in this paper does not allow one to make assertions about numeric-valued variables and numeric-valued fields. Our approach could be extended to support such capabilities using methods developed in work on abstract interpretation that combines canonical abstraction with numeric abstractions [21, 28].

7. Conclusion and Future Work

This paper showed how to create a semi-decision procedure for a fragment of separation logic. The fragment of separation logic that we use has empty-heap assertions ($\emp$), equalities ($x = y$), points-to assertions ($x \rightarrow y$), acyclic-list-segment assertions ($\ls(x, y)$), and their negations as literals; it provides the connectives $\ast, \circledast, \land, \lor$. We believe that this is an interesting fragment, in that it contains formulas for which existing approaches do not apply.

For each 3L formula $\varphi$, the procedure performs a bottom-up evaluation of the formula, using a particular shape-analysis interpretation: if the answer is the empty set of 3-valued structures, then $\varphi$ is unsatisfiable. Thus, the work reported in the paper supports the thesis that abstract-interpretation concepts can help in the design and implementation of decision procedures.

Moreover, if $\varphi$ is satisfiable, then the procedure reports a set of abstract models—i.e., a value in the canonical-abstraction domain that overapproximates $\llbracket \varphi \rrbracket$. As we have shown in other work (us-
ing a variety of other techniques, and for a variety of other logics), a decision-procedure-like method that is prepared to return such “residual” answers provides a way to generate sound abstract transformers automatically [32, 35, 37, 39]. In particular, when \( \phi \) specifies the transition relation between the pre-state and post-state of a concrete transformer \( \tau \), a residuating decision procedure provides a way to create a sound abstract transformer \( \tau^\sharp \) for \( \tau \), directly from a specification in logic of \( \tau^\sharp \)'s concrete semantics. Consequently, the work reported in the paper also supports the thesis that abstract-interpretation-based decision procedures provide much promise for automating the construction of program-analysis tools. Using our semi-decision procedure, we now have a way to create abstract transformers based on canonical-abstraction domains directly from a specification of the semantics of a language’s concrete transformers, written in SL.

Although TVLA and separation logic have both been applied to the problem of analyzing programs that manipulate linked data structures, there has been only rather limited crossover of ideas between the two approaches. Our semi-decision procedure is built on TVLA states and SL statelets described in §2.3, which represents the first formal connection between the two approaches. For this reason, the semi-decision procedure should be of interest to both communities: (i) For the TVLA community, the procedure illustrates a different and intriguing use for canonical-abstraction domains. The domains that we use are tailored for the particular formula, but, more importantly, provide an encoding that can be connected to the SL semantics: see Eqs. (2) and (3) in §2.3, and the use of domain predicates to express disjointness in §3. (ii) For the separation-logic community, the procedure shows how using TVLA and canonical-abstraction domains leads to a model-theoretic approach to the decision problem for SL that is capable of handling formulas that are beyond the capabilities of existing tools. We believe that the approach presented in this paper has the potential to be extended to deal with richer fragments of separation logic—in particular, fragments that contain both separating implication and acyclic linked-list predicates.

References


A. A Satisfiable Formula

Consider the formula \( \varphi \equiv x \rightarrow y \land \mathsf{ls}(x, z) \). We want to compute \( A \in \mathcal{A} \) such that \( \gamma(A) \models [\varphi] \). Similar to what was done in §3 for the \( \wedge \) operator, we introduce two new domain predicates \( d_1 \) and \( d_2 \), which are used to demarcate the heaplets that must satisfy \( \varphi_1 \equiv x \rightarrow y \) and \( \varphi_2 \equiv \mathsf{ls}(x, z) \). By design, there exist \( A_1, A_2 \in \mathcal{A} \) such that \( \gamma(A_1)\models [d_1] \models [x \rightarrow y] \) and \( \gamma(A_2)\models [d_2] \models [\mathsf{ls}(x, z)] \), respectively. \( A_1 \) consists of the single 3-valued structure shown in Fig. 6(b). Fig. 6(c) shows one of the structures in \( A_2 \); it represents an acyclic linked list from \( x \) to \( z \) whose length is greater than 1. Furthermore, to satisfy \( \varphi_1 \land \varphi_2 \), \( d_1 \) and \( d_2 \) are required to be disjoint regions whose union is \( d_2 \). \( A \) also contains an abstract value, which we will call \( D \), that represents this disjointness constraint exactly. \( D \) consists of four 3-valued structures. Fig. 6(a) shows the "most general" of them: it represents two disjoint regions, \( d \) and \( d_1 \), that partition the \( d_2 \) region (where each of \( d \) and \( d_1 \) contain at least one cell). The summary individual labeled \( \neg d, \neg d_1, \neg d_2 \) in Fig. 6(a) represents a region that is disjoint from \( d_2 \).

To impose a necessary condition for \( x \rightarrow y \land \mathsf{ls}(x, z) \) to be satisfiable, we take the meet of \( D, A_1, \) and \( A_2 \): \( x \rightarrow y \land \mathsf{ls}(x, z) \) \( \models \) \( D \cap \neg A_1 \cap \neg A_2 \). Fig. 6(d) shows one of the structures that arises in \( D \cap \neg A_1 \cap \neg A_2 \), after the semantic-reduction operators have been applied. A few points to note about this resultant structure:

- The summary individual in region \( d_2 \) present in the \( \mathsf{ls}(x, z) \) structure in Fig. 6(c) is split in Fig. 6(d) into a singleton individual pointed to by \( y \) and a summary individual.
- The individual pointed to by \( x \) is in regions \( d_1 \) and \( d_2 \), but not \( d \).
- The individual pointed to by \( y \) is in regions \( d \) and \( d_2 \), but not \( d_1 \).
- The variables \( x \) and \( y \) are not equal.
- All the individuals in \( d \) are reachable from \( y \), not reachable from \( z \), and have \( \text{link}[d, n, z] \) true.

Fig. 6(e) shows the structure after we have projected the heap onto the heap region \( d \); that is, the values of the domain predicates \( d_1 \) and \( d_2 \) have been set of 1/2 on all individuals, and all the abstraction predicates have been set to 1/2 on all individuals not in \( d \). In effect, this operation blurs the distinction between the region that is outside \( d \), but in \( d_2 \), and the region that is outside of \( d \) and \( d_2 \). Note that the fact that \( x \) and \( y \) are not equal is preserved by the projection operation. This projection operation, denoted by \( (\cdot)^{\downarrow}_d \) in §4, serves as an abstract method for quantifier elimination.

Note that Fig. 6(e) represents an acyclic linked-list from \( y \) to \( z \) with \( x \neq y \), which is one of the models that satisfies \( x \rightarrow y \land \mathsf{ls}(x, z) \).

Figure 6: Some of the structures that arise in the meet operation used to evaluate \( x \rightarrow y \land \mathsf{ls}(x, z) \).